The Mondrian Kernel

Matej Balog

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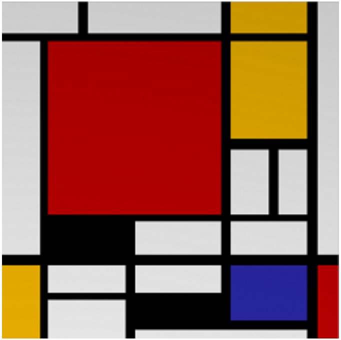
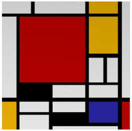
Daniel M. Roy

Yee Whye Teh



Mondrian
kernel

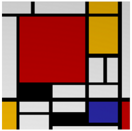
$$k(\cdot, \cdot) \ : \ \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$$



Mondrian
process

↓

Mondrian
kernel

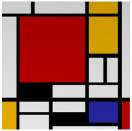


Mondrian process

↓

Mondrian kernel ⟶ Laplace kernel

$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x} - \mathbf{x}'\|_1}$$

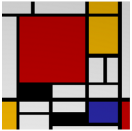


Mondrian process

Mondrian kernel

Laplace kernel

$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x} - \mathbf{x}'\|_1}$$



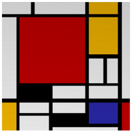
Mondrian
process

Mondrian
kernel

Laplace
kernel

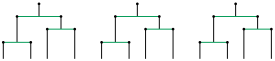
$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$$

inverse width



Mondrian process

Mondrian forest

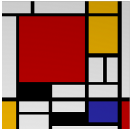


Mondrian kernel

Laplace kernel

$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$$

inverse width (↑ λ)

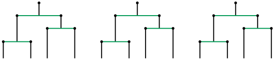


Mondrian process

Mondrian forest

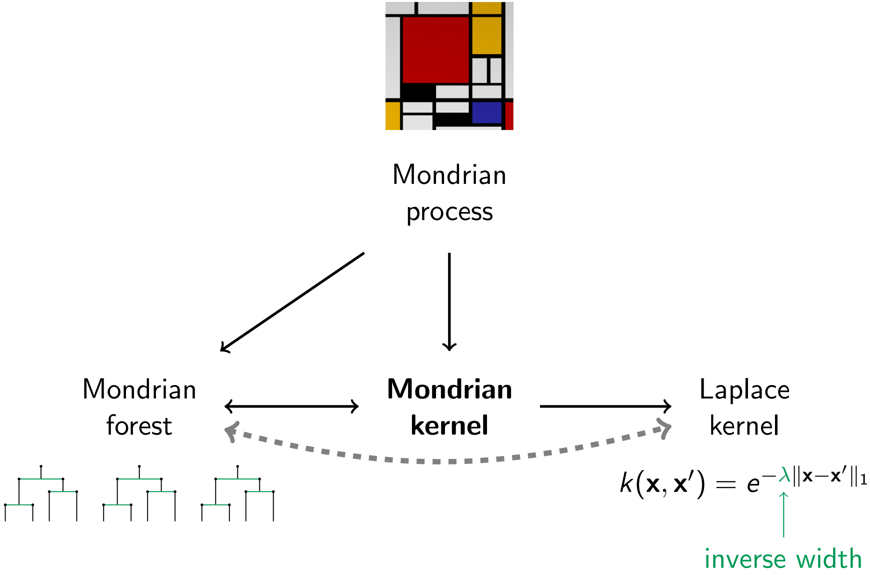
Mondrian kernel

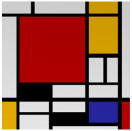
Laplace kernel



$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$$

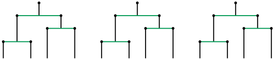
inverse width


Mondrian process
Mondrian forest
Mondrian kernel
Laplace kernel
$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$
inverse width



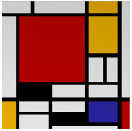
1 Mondrian process

Mondrian forest ⟷ **Mondrian kernel** ⟶ Laplace kernel



$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$$

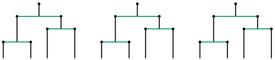
λ: inverse width



1 Mondrian process

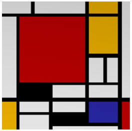
2 **Mondrian kernel**

Mondrian forest ⟷ **Mondrian kernel** ⟶ Laplace kernel



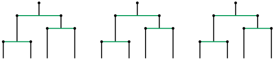
$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$$

λ: inverse width



1 Mondrian process

Mondrian forest



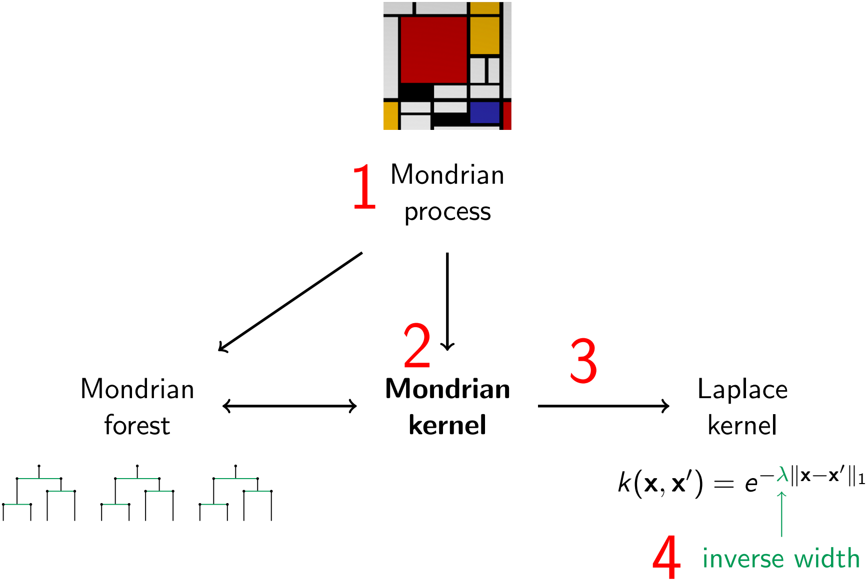
2 **Mondrian kernel**

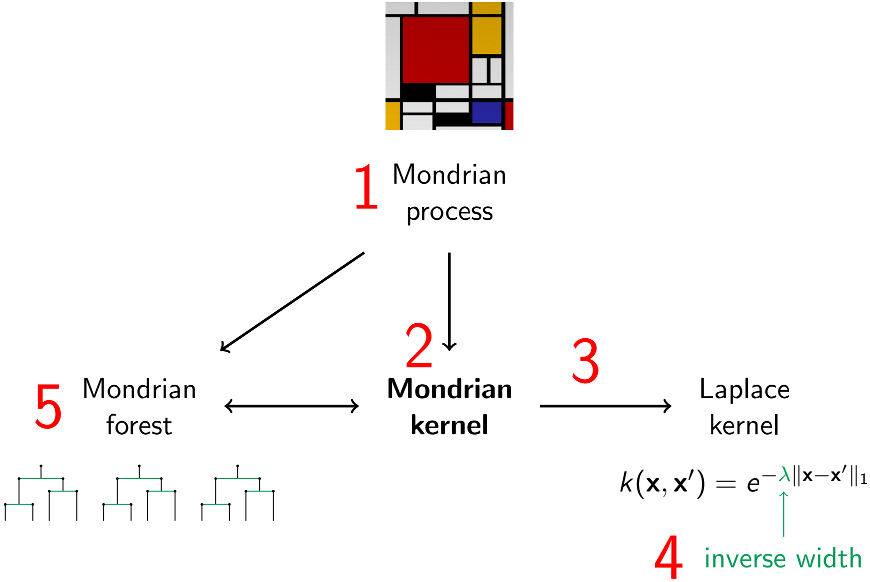
3

Laplace kernel

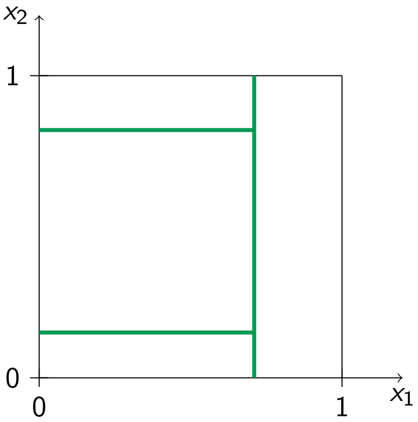
$$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x} - \mathbf{x}'\|_1}$$

λ: inverse width

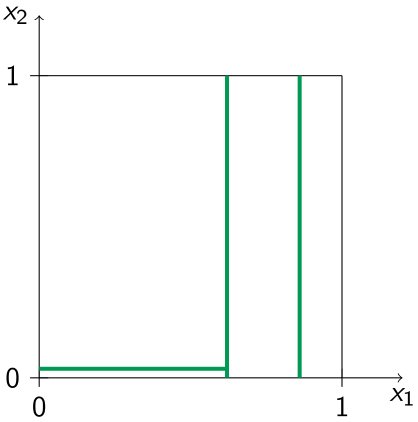

1 Mondrian process
2 Mondrian kernel
3
Mondrian forest
Laplace kernel
$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$
4 inverse width


1 Mondrian process
2 Mondrian kernel
3
Laplace kernel
$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$
4 inverse width
5 Mondrian forest

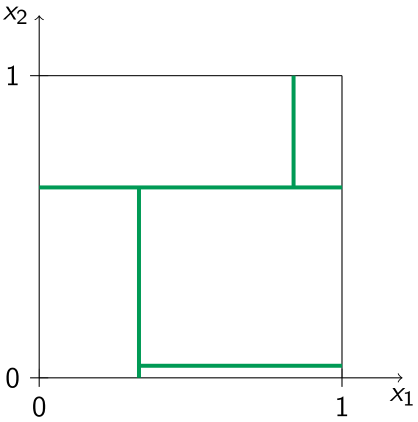
Mondrian Process



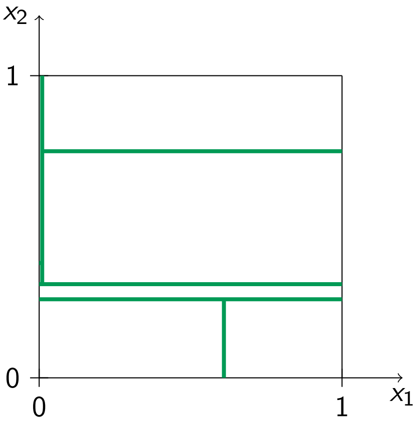
Mondrian Process



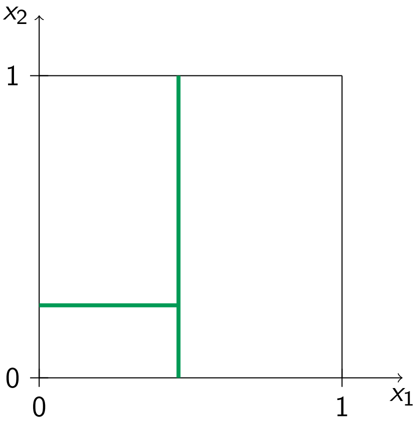
Mondrian Process



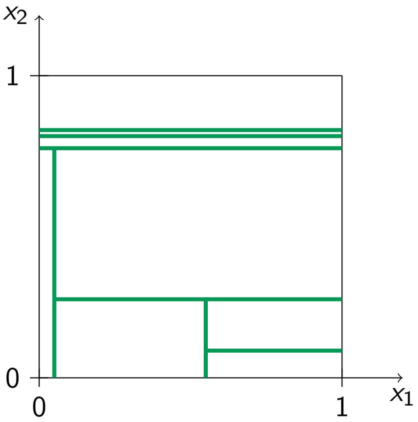
Mondrian Process



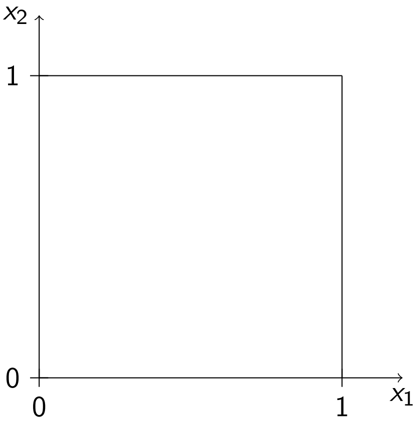
Mondrian Process



Mondrian Process



Mondrian Process



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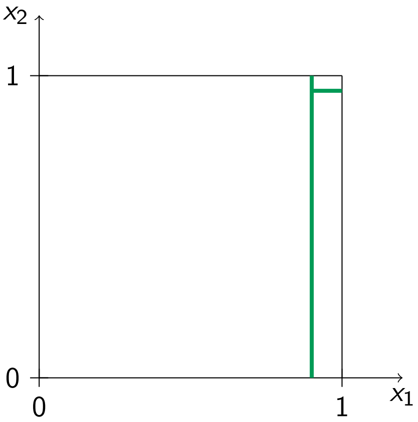

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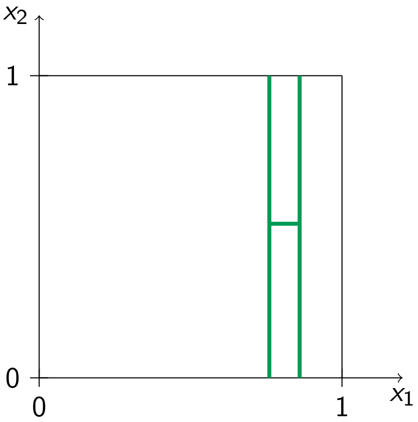

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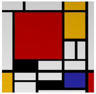

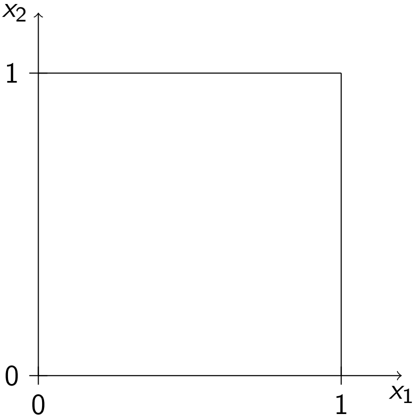

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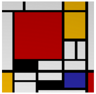

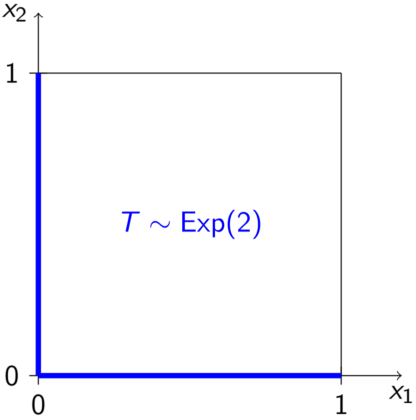

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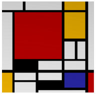

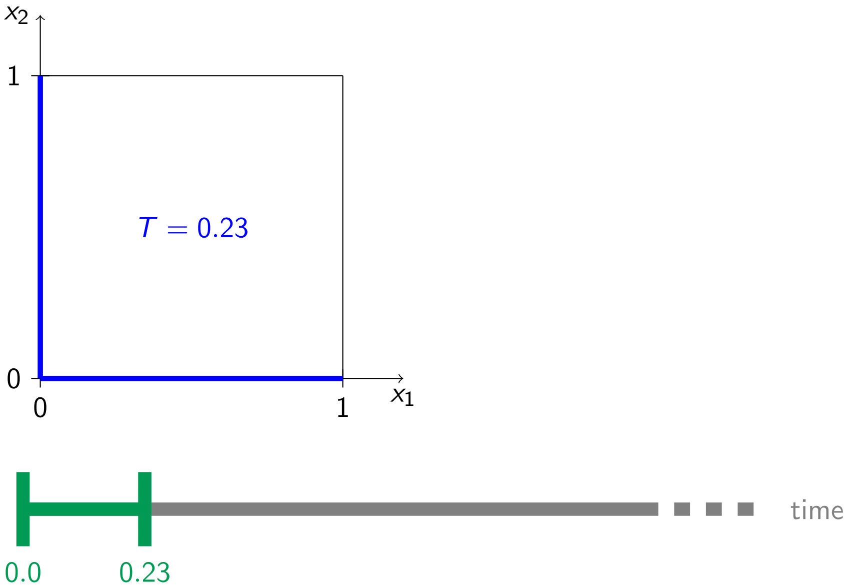

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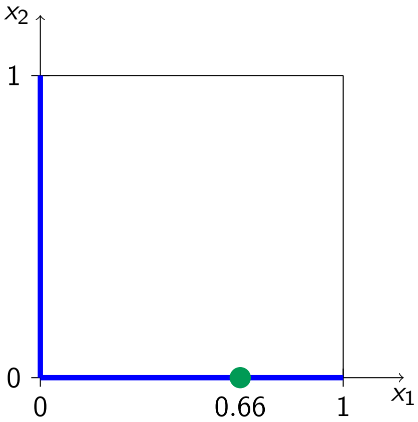

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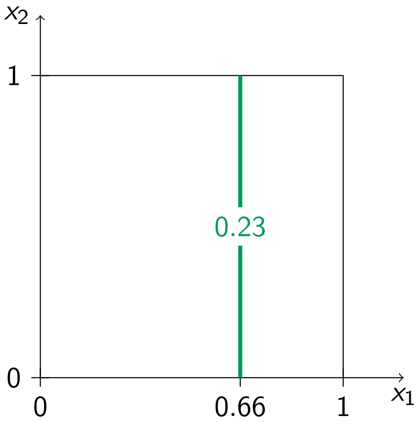

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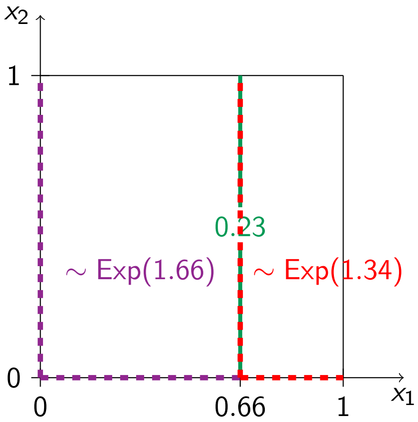

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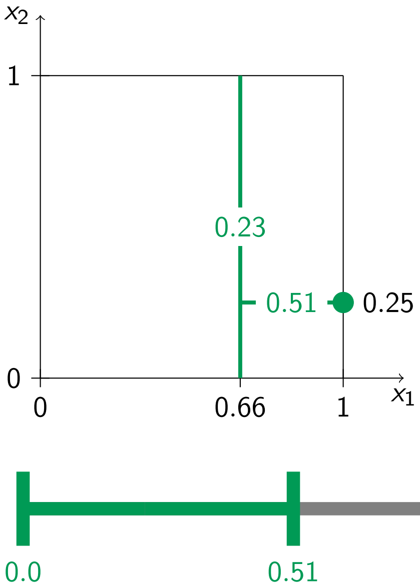

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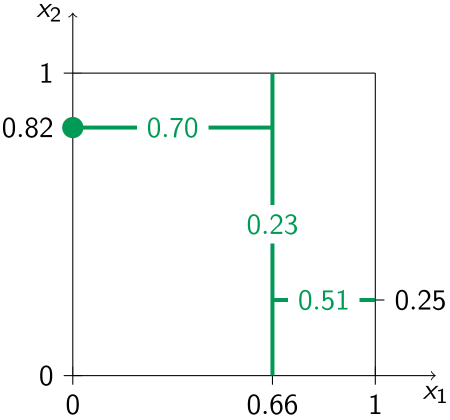

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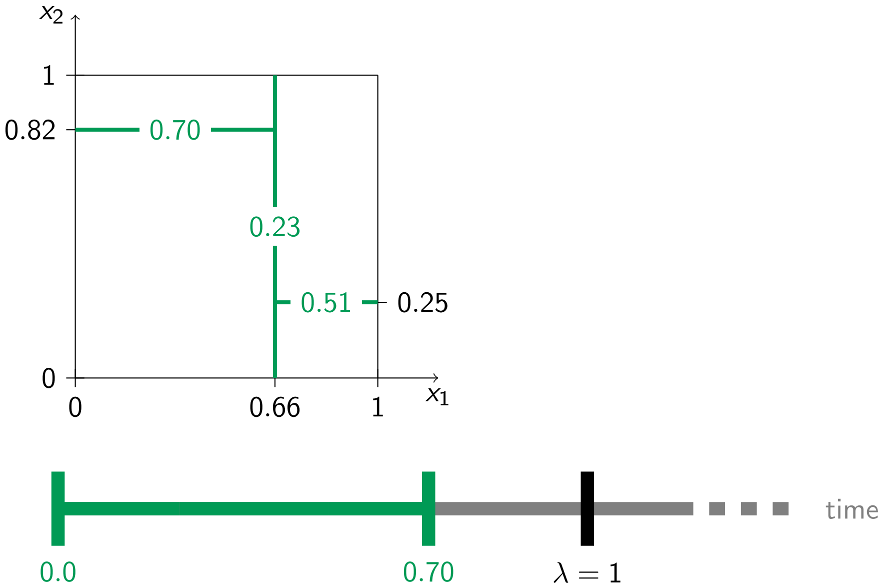

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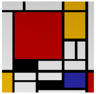

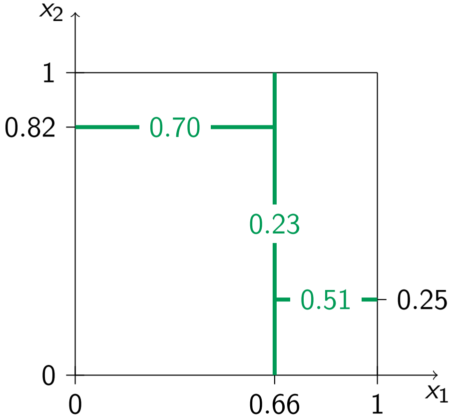

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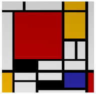
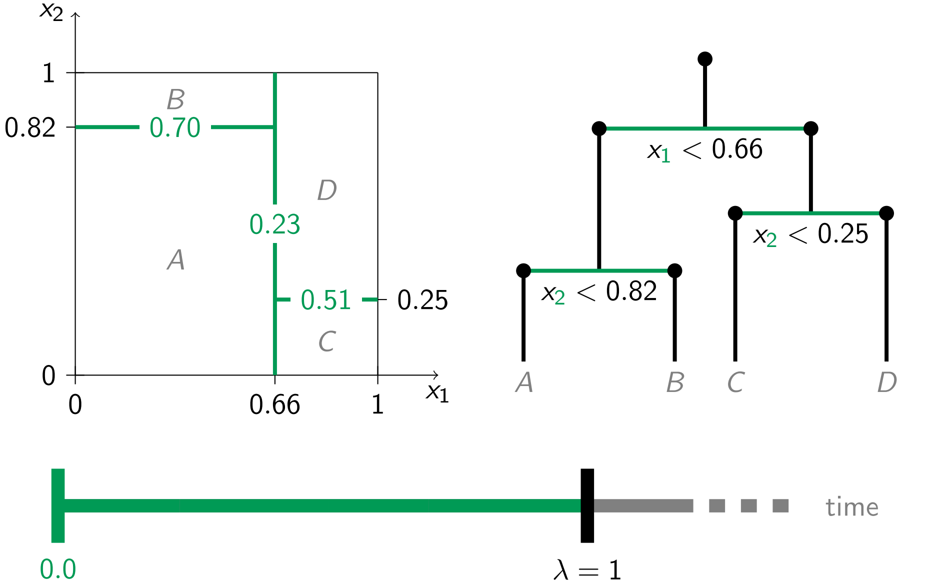
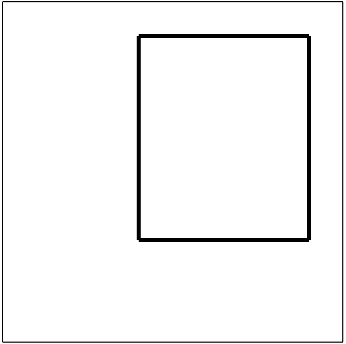
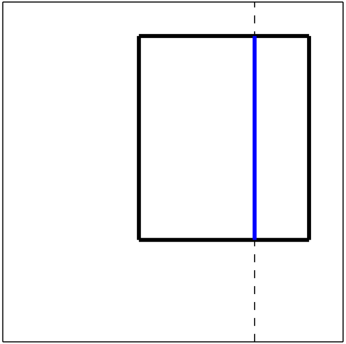




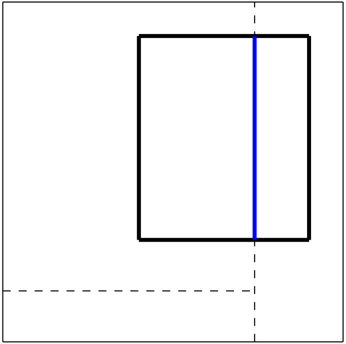
Mondrian Process – projectivity



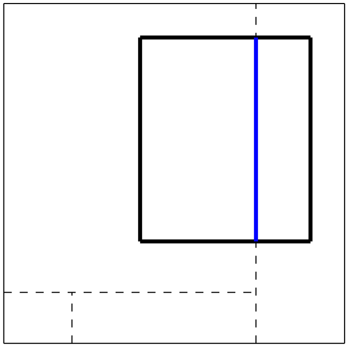
Mondrian Process – projectivity



Mondrian Process – projectivity



Mondrian Process – projectivity



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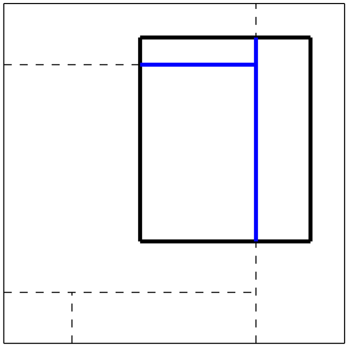

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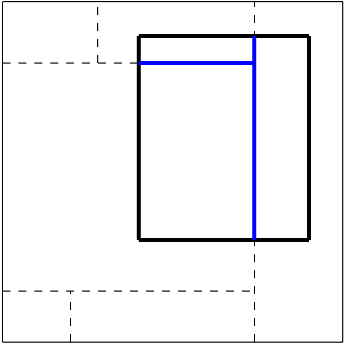

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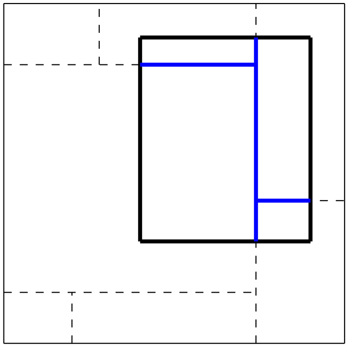
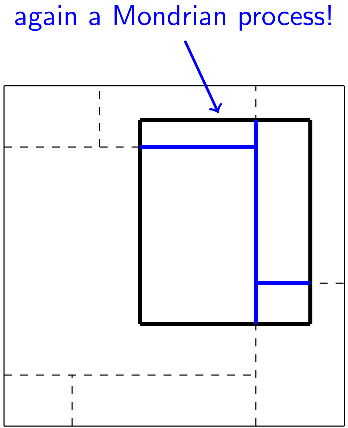
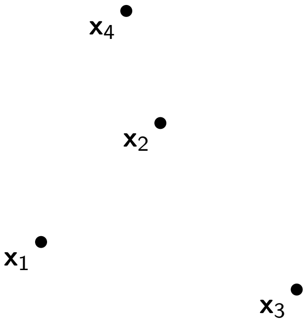


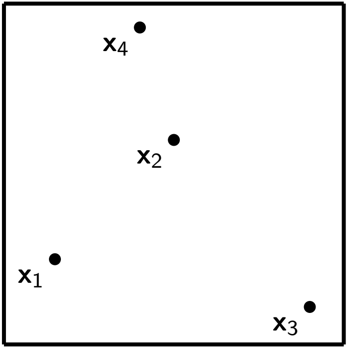
Mondrian Process – projectivity



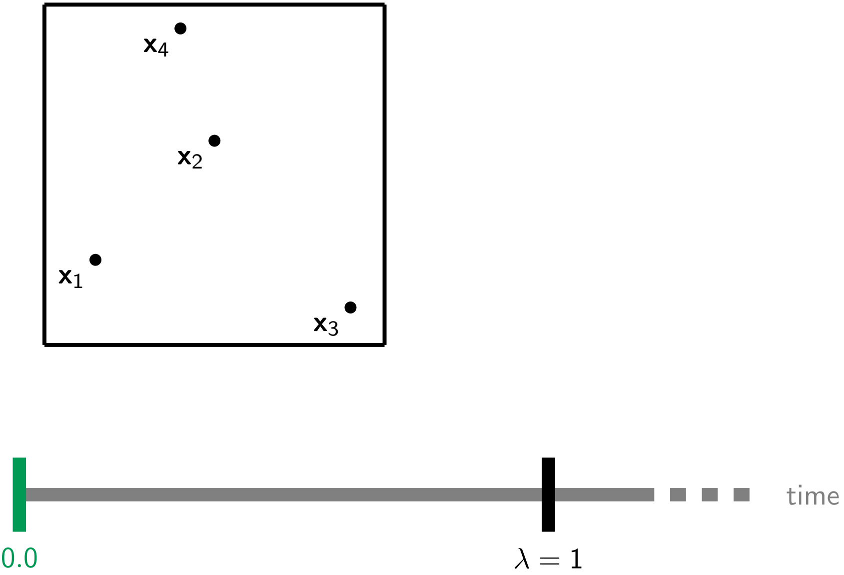
Mondrian kernel



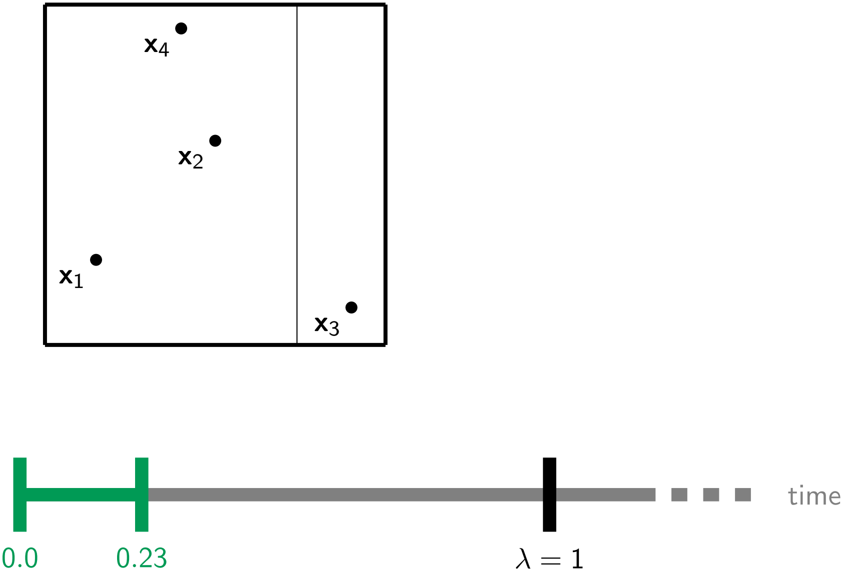
Mondrian kernel



Mondrian kernel



Mondrian kernel



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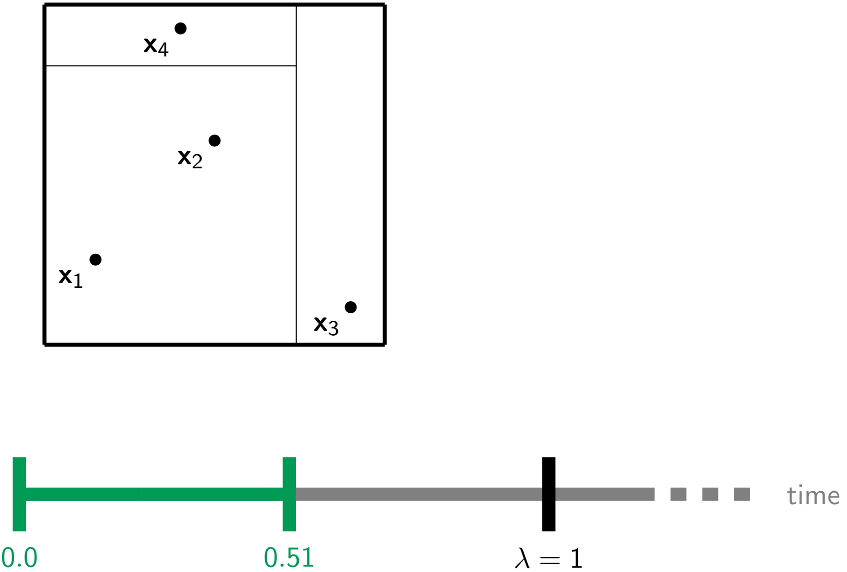

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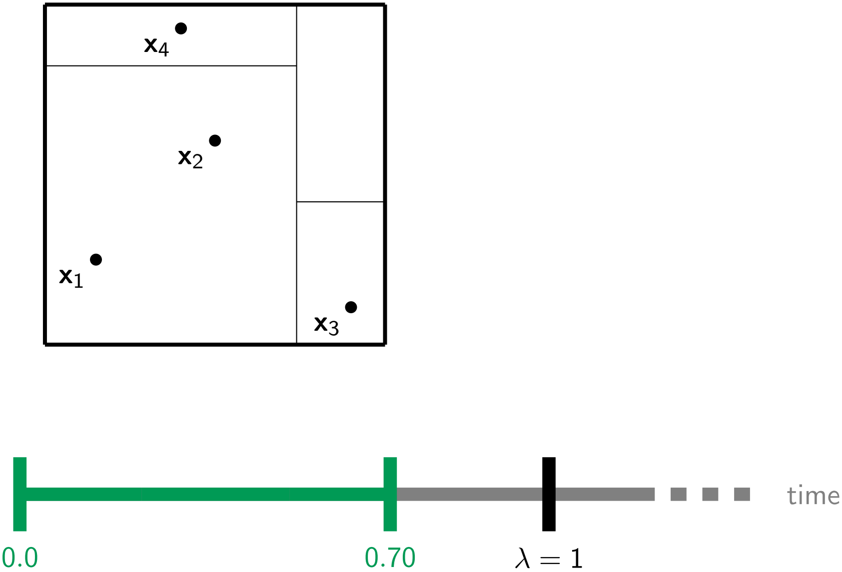
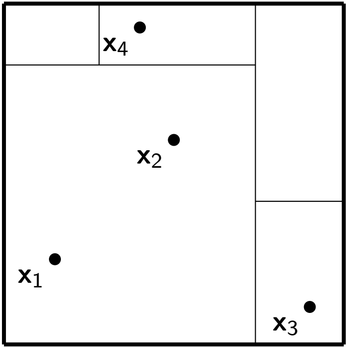


Mondrian kernel



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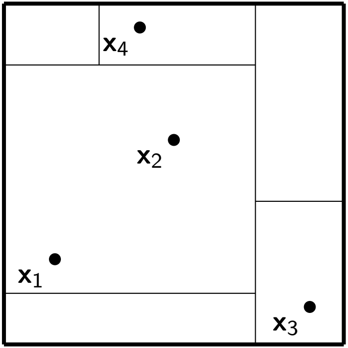
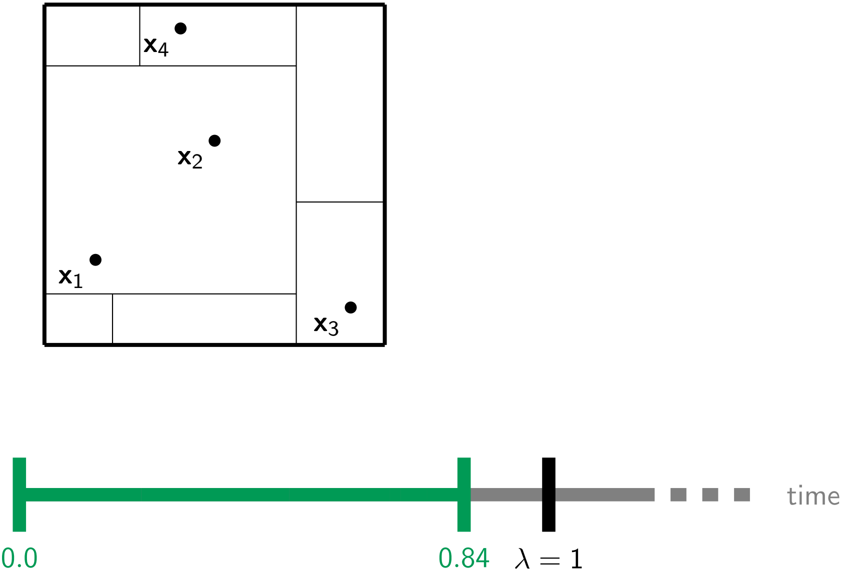
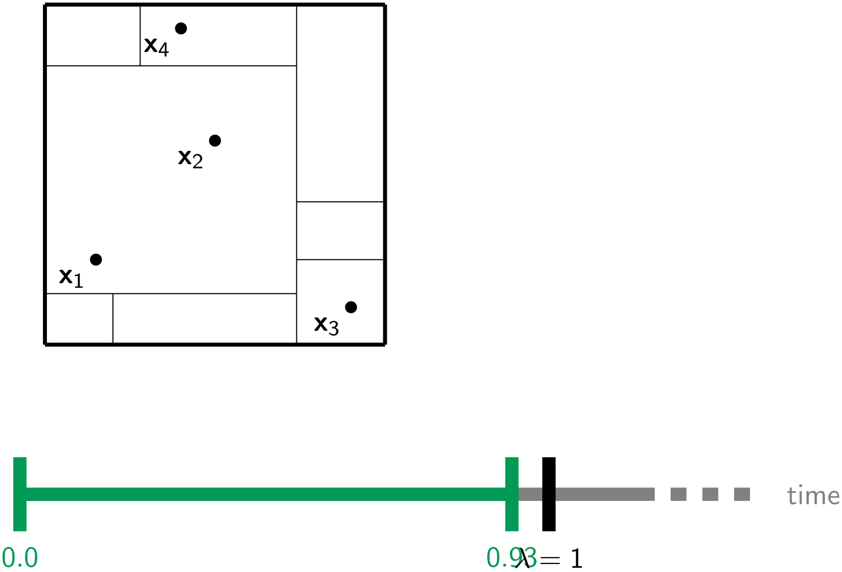


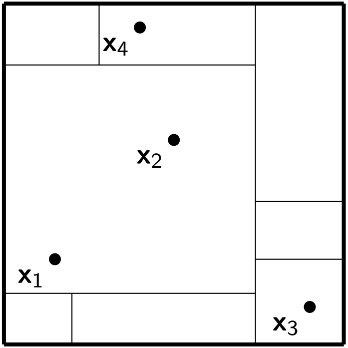
Mondrian kernel



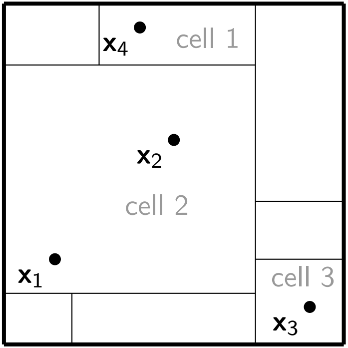
Mondrian kernel



Mondrian kernel

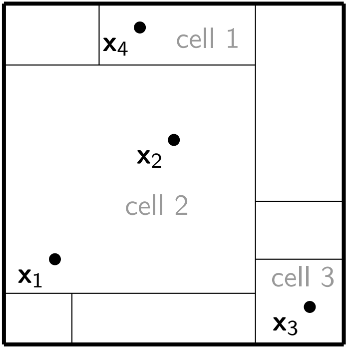


Mondrian kernel



$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 1 0]
$\mathbf{x}_2$	[0 1 0]
$\mathbf{x}_3$	[0 0 1]
$\mathbf{x}_4$	[1 0 0]

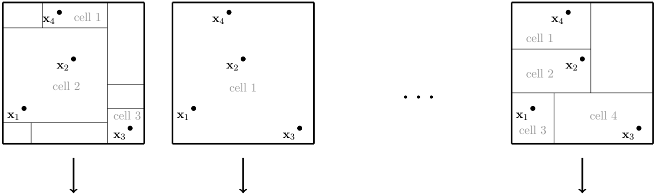
Mondrian kernel (of order 1)



$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 1 0]
$\mathbf{x}_2$	[0 1 0]
$\mathbf{x}_3$	[0 0 1]
$\mathbf{x}_4$	[1 0 0]

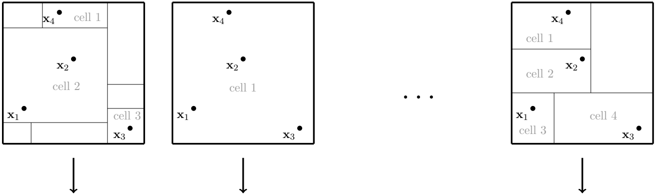
$$k_1(\mathbf{x}, \mathbf{x}') := \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle = \begin{cases} 1 & \text{if } \mathbf{x}, \mathbf{x}' \text{ in the same cell} \\ 0 & \text{otherwise} \end{cases}$$

Mondrian kernel (of order M)



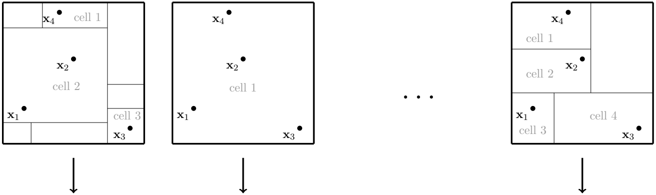
[0 1 0]	[1]	$\cdots$	[0 0 1 0]
[0 1 0]	[1]	$\cdots$	[0 1 0 0]
[0 0 1]	[1]	$\cdots$	[0 0 0 1]
[1 0 0]	[1]	$\cdots$	[1 0 0 0]

Mondrian kernel (of order M)



$$\phi(\mathbf{x}_1) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \quad 1 \quad \cdots \quad 0\ 0\ 1\ 0 \quad]$$
$$\phi(\mathbf{x}_2) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \quad 1 \quad \cdots \quad 0\ 1\ 0\ 0 \quad]$$
$$\phi(\mathbf{x}_3) = \frac{1}{\sqrt{M}} [\quad 0\ 0\ 1 \quad 1 \quad \cdots \quad 0\ 0\ 0\ 1 \quad]$$
$$\phi(\mathbf{x}_4) = \frac{1}{\sqrt{M}} [\quad 1\ 0\ 0 \quad 1 \quad \cdots \quad 1\ 0\ 0\ 0 \quad]$$

Mondrian kernel (of order M)



$$\phi(\mathbf{x}_1) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \qquad 1 \qquad \cdots \qquad 0\ 0\ 1\ 0 \quad]$$

$$\phi(\mathbf{x}_2) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \qquad 1 \qquad \cdots \qquad 0\ 1\ 0\ 0 \quad]$$

$$\phi(\mathbf{x}_3) = \frac{1}{\sqrt{M}} [\quad 0\ 0\ 1 \qquad 1 \qquad \cdots \qquad 0\ 0\ 0\ 1 \quad]$$

$$\phi(\mathbf{x}_4) = \frac{1}{\sqrt{M}} [\quad 1\ 0\ 0 \qquad 1 \qquad \cdots \qquad 1\ 0\ 0\ 0 \quad]$$

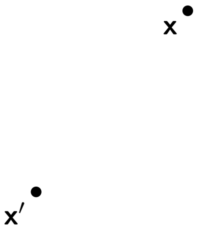
$$k_M(\mathbf{x}, \mathbf{x}') := \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell of } m\text{-th partition}\}}$$

Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$

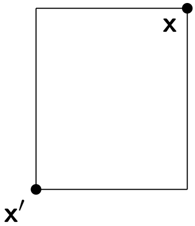
Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$



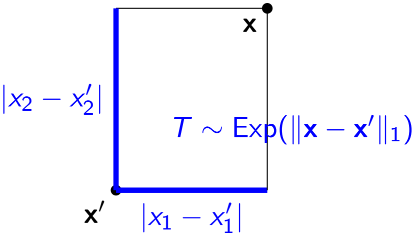
Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$



Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$

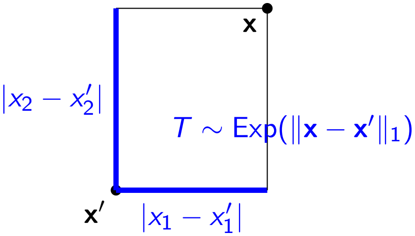


Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x},\mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$

$$\parallel$$

$$\mathbb{P}\left(T > \lambda\right) \text{ where } T \sim \operatorname{Exp}\left(\|\mathbf{x} - \mathbf{x}'\|_1\right)$$



Mondrian-Laplace connection

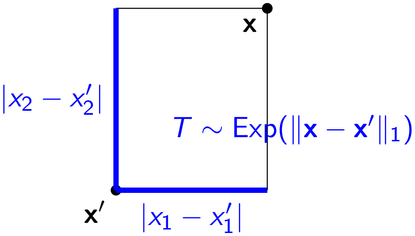
$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x},\mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$

$$||$$

$$\mathbb{P}\left(T > \lambda\right) \text{ where } T \sim \text{Exp}\left(\|\mathbf{x} - \mathbf{x}'\|_1\right)$$

$$||$$

$$\exp\left(-\lambda \|\mathbf{x} - \mathbf{x}'\|_1\right)$$



Mondrian-Laplace connection

$$k_M(\mathbf{x}, \mathbf{x}') = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x}, \mathbf{x}' \text{ in same cell } | m\}} \xrightarrow{M \to \infty} \mathbb{P}\left[\mathbf{x}, \mathbf{x}' \text{ in same cell }\right]$$

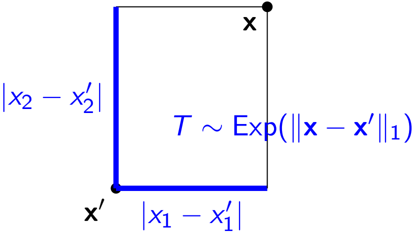
$$||$$

$$\mathbb{P}\left(T > \lambda\right) \text{ where } T \sim \text{Exp}\left(\|\mathbf{x} - \mathbf{x}'\|_1\right)$$

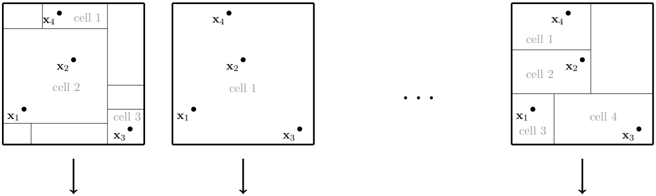
$$||$$

$$\exp\left(-\lambda \|\mathbf{x} - \mathbf{x}'\|_1\right)$$

Mondrian process lifetime



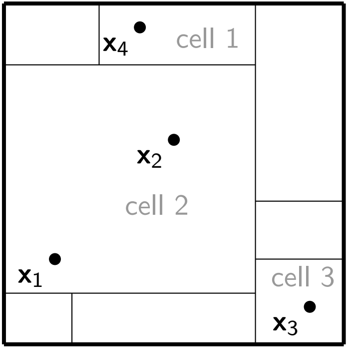
Mondrian kernel (of order M)



$$\phi(\mathbf{x}_1) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \qquad 1 \qquad \cdots \qquad 0\ 0\ 1\ 0 \quad]$$
$$\phi(\mathbf{x}_2) = \frac{1}{\sqrt{M}} [\quad 0\ 1\ 0 \qquad 1 \qquad \cdots \qquad 0\ 1\ 0\ 0 \quad]$$
$$\phi(\mathbf{x}_3) = \frac{1}{\sqrt{M}} [\quad 0\ 0\ 1 \qquad 1 \qquad \cdots \qquad 0\ 0\ 0\ 1 \quad]$$
$$\phi(\mathbf{x}_4) = \frac{1}{\sqrt{M}} [\quad 1\ 0\ 0 \qquad 1 \qquad \cdots \qquad 1\ 0\ 0\ 0 \quad]$$

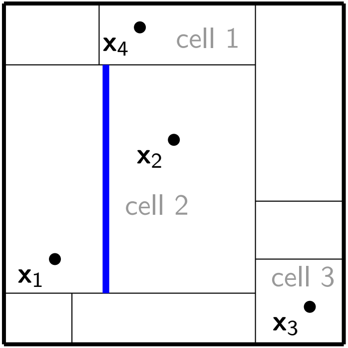
$$k_M(\mathbf{x}, \mathbf{x}') := \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{\mathbf{x},\mathbf{x}' \text{ in same cell of } m\text{-th partition}\}}$$
$$\xrightarrow{M \to \infty} \exp\left(-\lambda \|\mathbf{x} - \mathbf{x}'\|_1\right)$$

Kernel width selection



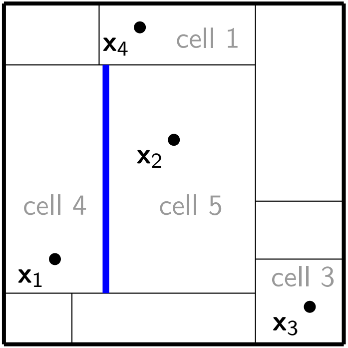
$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 1 0]
$\mathbf{x}_2$	[0 1 0]
$\mathbf{x}_3$	[0 0 1]
$\mathbf{x}_4$	[1 0 0]

Kernel width selection



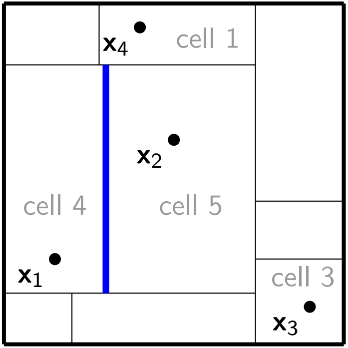
$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 1 0]
$\mathbf{x}_2$	[0 1 0]
$\mathbf{x}_3$	[0 0 1]
$\mathbf{x}_4$	[1 0 0]

Kernel width selection



$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 *1* 0 **1** **0**]
$\mathbf{x}_2$	[0 *1* 0 **0** **1**]
$\mathbf{x}_3$	[0 *0* 1 **0** **0**]
$\mathbf{x}_4$	[1 *0* 0 **0** **0**]

Kernel width selection



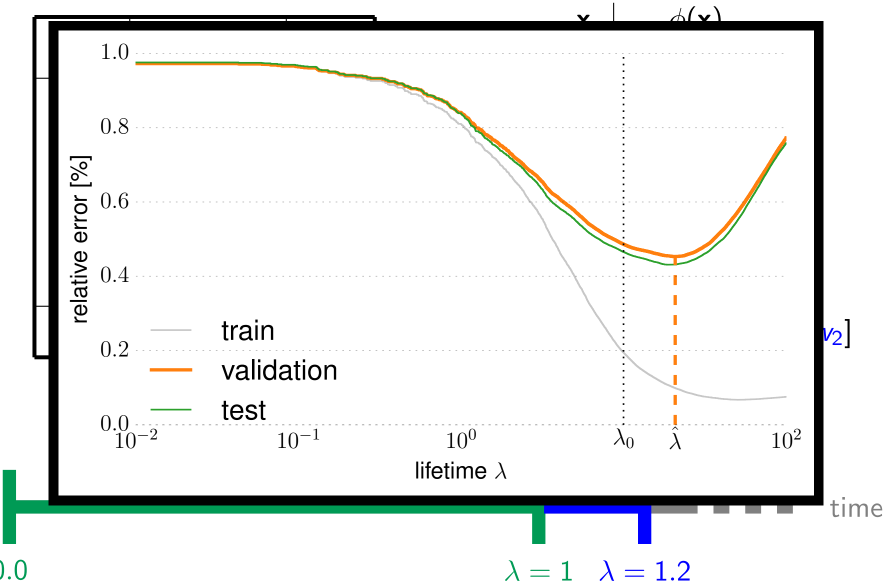
$\mathbf{x}$	$\phi(\mathbf{x})$
$\mathbf{x}_1$	[0 *1* 0 **1** **0**]
$\mathbf{x}_2$	[0 *1* 0 **0** **1**]
$\mathbf{x}_3$	[0 *0* 1 **0** **0**]
$\mathbf{x}_4$	[1 *0* 0 **0** **0**]

$$\mathbf{w}_{\text{opt}} = [w_1 \;\; w_2 \;\; w_3]$$

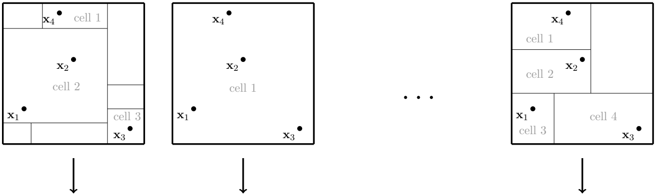
$$\downarrow$$

$$\mathbf{w}_{\text{init}} = [w_1 \;\; \cancel{w_2} \;\; w_3 \;\; w_2 \;\; w_2]$$

Kernel width selection



Mondrian kernel vs Mondrian forest

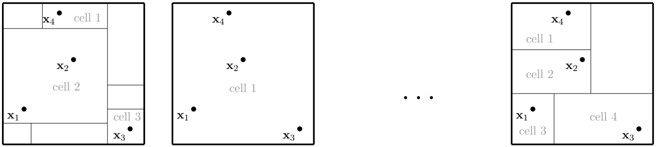


$$\phi(\mathbf{x}_1) = \frac{1}{\sqrt{M}} \big[\quad 0\,1\,0 \quad 1 \quad \cdots \quad 0\,0\,0\,1 \quad \big]$$
$$\phi(\mathbf{x}_2) = \frac{1}{\sqrt{M}} \big[\quad 0\,1\,0 \quad 1 \quad \cdots \quad 0\,1\,0\,0 \quad \big]$$
$$\phi(\mathbf{x}_3) = \frac{1}{\sqrt{M}} \big[\quad 0\,0\,1 \quad 1 \quad \cdots \quad 0\,0\,1\,0 \quad \big]$$
$$\phi(\mathbf{x}_4) = \frac{1}{\sqrt{M}} \big[\quad 1\,0\,0 \quad 1 \quad \cdots \quad 1\,0\,0\,0 \quad \big]$$

kernel

feature weights fit jointly: $\min_{\mathbf{w}} \|\mathbf{\Phi}\mathbf{w} - \mathbf{y}\|_2^2$

Mondrian kernel vs Mondrian forest



$$\phi(\mathbf{x}_1) = \frac{1}{\sqrt{M}}\left[\; 0\,1\,0 \quad 1 \quad \cdots \quad 0\,0\,0\,1 \;\right]$$

$$\phi(\mathbf{x}_2) = \frac{1}{\sqrt{M}}\left[\; 0\,1\,0 \quad 1 \quad \cdots \quad 0\,1\,0\,0 \;\right]$$

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$$\phi(\mathbf{x}_4) = \frac{1}{\sqrt{M}}\left[\; 1\,0\,0 \quad 1 \quad \cdots \quad 1\,0\,0\,0 \;\right]$$

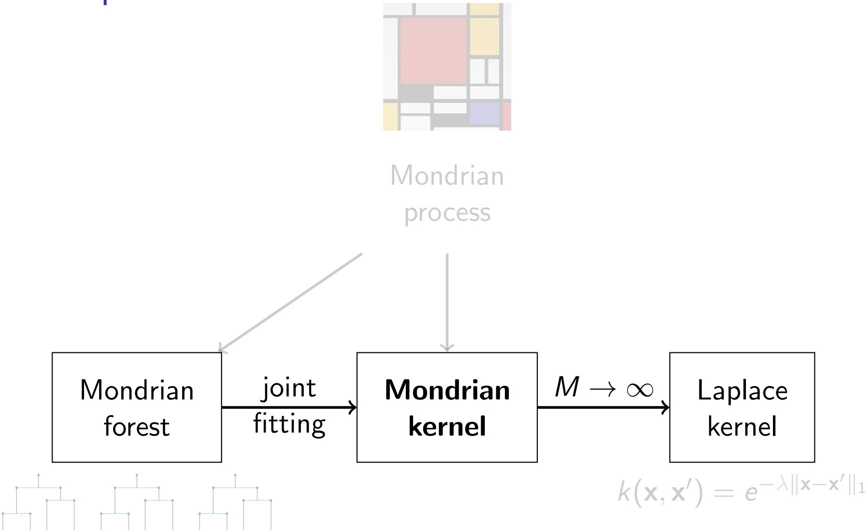
kernel

feature weights fit jointly: $\min_{\mathbf{w}} \|\mathbf{\Phi}\mathbf{w} - \mathbf{y}\|_2^2$

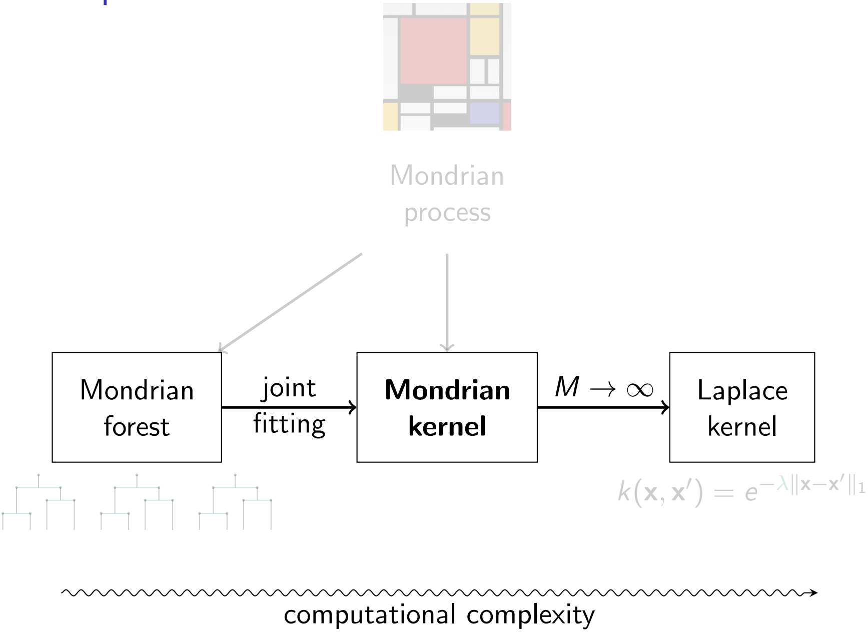
forest

fit trees independently

Landscape


Mondrian process
Mondrian forest
joint fitting
Mondrian kernel
$M \to \infty$
Laplace kernel
$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$

Landscape


Mondrian process
Mondrian forest
joint fitting
Mondrian kernel
$M \to \infty$
Laplace kernel
$k(\mathbf{x}, \mathbf{x}') = e^{-\lambda \|\mathbf{x}-\mathbf{x}'\|_1}$
computational complexity

Thank you

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